

ON SUBGROUPS OF THE ADDITIVE GROUP
IN DIFFERENTIALLY CLOSED FIELDS

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Abstract. In this paper we deal with the model theory of differentially closed fields of characteristic zero with finitely many commuting derivations. First we observe that the only known lower bound for the Lascar rank of types in differentially closed fields, announced in a paper of McGrail, is false. This gives us a new class of regular types which are orthogonal to fields. Then we classify the subgroups of the additive group of Lascar rank ω with differential-type 1 which are nonorthogonal to fields. The last parts consist of an analysis of the quotients of the heat variety. We show that the generic type of such a quotient is locally modular. Finally, we prove that a conjecture of Phyllis Cassidy about the existence of certain Jordan-Hölder type series is false.

§1. Introduction. The model theoretic study of ordinary differential fields began with Robinson, who, in [16], introduced the notion of differentially closed field. The next important progress in the area came from Blum who gave an elegant axiomatization of the theory of ordinary differentially closed fields. After Blum, the interaction between model theory and differential algebra grew rapidly. Differential algebra kept providing interesting examples for model theorists and model theoretic methods solved some problems posed by differential algebraists. The most remarkable of these interactions was, of course, Hrushovski's proof of the Mordell-Lang theorem, which, in characteristic zero, uses Zilber's dichotomy for ordinary differentially closed fields.

However, despite the success of model theoretic approach to the study of ordinary differential fields, there have not been many attempts to do the same thing for partial differential fields. The first paper in this direction was McGrail's [8], which generalized the basic results on ordinary differential fields, such as ω -stability, to partial differential fields. The first attempt to prove Zilber's dichotomy in the partial differential case was [9]. In that paper Moosa, Pillay and Scanlon proved that a non-locally modular regular type is nonorthogonal to the generic type of a subgroup of the

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additive group. This result, which reduces the problem to the analysis of definable subgroups of the additive group, is the main motivation behind this paper.

§2. Differential fields. We will assume that the reader is familiar with the basic notions of differential algebra and differential algebraic geometry. All the theorems we will use can be found in [7] and [6]. Everything will take place in a big saturated model \mathcal{U} of DCF_{0m} . Our notation is pretty standard and consistent with our references. The only exception is the following. If K is a differential field and $\sigma: K \rightarrow K$ is any function, we define a new function from $K\{y\}$ to $K\{y\}$, denoted by $f \mapsto f^\sigma$, by applying σ to the coefficients of f .

The basic algebraic tool we need is the dimension polynomial.

THEOREM 2.1. *(Theorem 6 in section 2.6 of [6]) Let K be a Δ -field with $|\Delta| = m$ and let η be a finite tuple from an extension of K . Then there is a numerical polynomial $\omega_{\eta/K}$, called the dimension polynomial of η over K , such that*

- For large enough s , $\omega_{\eta/K}(s)$ is equal to the transcendence degree of $K(\theta\eta: \theta \in \Theta(s))$ over K .
- If $\omega_{\eta/K}(X) = \sum_{i=0}^m a_i \binom{X+i}{i}$, $a_0, \dots, a_m \in \mathbb{Z}$, then a_m , which may be zero, is the Δ -transcendence degree of $K(\eta)$ over K .

It is worth noting that the dimension polynomial is not a Δ -birational invariant, however, its degree and leading coefficient are. If

$$\omega_{\eta/K}(X) = \sum_{i=0}^{\tau} a_i \binom{X+i}{i}, \quad a_\tau \neq 0,$$

then τ is called the Δ -type of η over K and a_τ is called the *typical Δ -dimension* of η over K . From now on we will write all dimension polynomials with respect to this basis.

We define the dimension polynomial of an irreducible Δ -closed set to be the dimension polynomial of a generic point of the set. For an irreducible variety X , we will denote the Δ -type of X by $\tau(X)$ and the i -th coefficient of the dimension polynomial of X by $a_i(X)$. So, if $\tau = \tau(X)$, then $a_\tau(X)$ is the typical Δ -dimension of X . In the case of connected Δ -groups, τ and a_τ behave particularly well.

THEOREM 2.2. *(Theorem 1 in section 4.2 of [7]) Let G and H be connected Δ -groups and let φ be a definable group homomorphism from G onto H with kernel H' . Set $\tau = \tau(G)$. Then*

$$\tau = \max\{\tau(H'), \tau(H)\} \quad \text{and} \quad a_\tau(G) = a_\tau(H) + a_\tau(H').$$

The following notion was first defined by Kolchin and Cassidy under the name *almost Δ -simplicity*. Here we adopt the name used in [9].

DEFINITION 2.3. Let G be a Δ -group whose defining differential ideal over \mathcal{U} is prime. We say that G is Δ -type minimal if it does not have a proper Δ -subgroup H with $\tau(H) = \tau(G)$.

Let G be a Δ -type minimal Δ -group. Clearly G is connected and if H is a nontrivial quotient of G , then, by the previous theorem, $\tau(G) = \tau(H)$ and $a_\tau(G) = a_\tau(H)$. Moreover, by corollary 3.10 of [9], if G is a nontrivial subgroup of \mathbb{G}_a^n for some n then the generic type of G is regular.

The dimension polynomial of a variety can be computed if we know a characteristic set of its defining ideal with respect to an orderly ranking. For a subset E of \mathbb{N}^m , which is viewed as a lattice under the product order, we let V_E be the set of elements in \mathbb{N}^m which are not greater than or equal to any element of E . Then, one can show that the function defined by

$$\omega_E(s) = \#\{(e_1, \dots, e_m) \in V_E : e_1 + \dots + e_m \leq s\}$$

is a numerical polynomial.

Now let \mathfrak{p} be a prime ideal in $K\{y\}$ with a generic zero η . We will give a formula for the dimension polynomial of η over K .

THEOREM 2.4. (*Theorem 6 in section 2.6 of [6]*) Suppose that Λ is a characteristic set of \mathfrak{p} with respect to an orderly ranking. Then $\omega_{\eta/K} = \sum_{i=1}^n \omega_{E_i}$ where

$$E_i = \{(e_1, \dots, e_m) \in \mathbb{N}^m : \delta_1^{e_1} \cdots \delta_m^{e_m} y_i \text{ is the leader of an element of } \Lambda\}.$$

In the rest of this section we will focus on the subgroups of the additive group. Most of this material is from [7], [2] and [18]. Our first theorem gives a characterization of subgroups of \mathbb{G}_a^n together with some extra information about the defining ideals.

THEOREM 2.5. (*See section of 2.4 of [2] and remark 2.1 of [17]*) A Kolchin-closed subset G of \mathbb{G}_a^n is a Δ -subgroup of \mathbb{G}_a^n if and only if the defining differential ideal \mathfrak{p} of G is generated, as a differential ideal, by a finite set of linear homogeneous differential polynomials. If this is the case, one can choose a characteristic set of \mathfrak{p} consisting of linear homogeneous polynomials. Moreover, if Λ is such a characteristic set, a monomial of the form θy_i is the leader of a linear homogeneous polynomial in \mathfrak{p} if and only if there is an element in Λ with leader $\theta_2 y_i$ such that $\theta y_i = \theta_1 \theta_2 y_i$ for some θ_1 in Θ .

The next theorem sometimes makes it very easy to find characteristic sets for linear differential ideals. We will use it in the section on rank inequalities.

THEOREM 2.6. (*See section of 2.4 of [2]*) If Λ is an autoreduced and coherent set consisting of homogeneous linear differential polynomials, then it is a characteristic set of $[\Lambda]$.

As one can expect, definable homomorphisms between these groups are given by linear homogeneous polynomials. However, the proof of this fact is not as straightforward as the purely algebraic case. Cassidy, in [3], solves this problem for a class of groups which extends the class of subgroups of \mathbb{G}_a^n .

THEOREM 2.7. *Let G and H be definable subgroups of \mathbb{G}_a^n for some n . A definable group homomorphism from G to H can be expressed by a linear homogeneous differential polynomial.*

The following proposition is standard.

PROPOSITION 2.8. *Let G_1 and G_2 be Δ -subgroups of \mathbb{G}_a and let X be an irreducible Δ -closed subset of $G_1 \times G_2$. If the projection map $\pi: X \rightarrow G_1$ is dominant, then there is a dense open subset U of X such that for all $x \in U$ the map $d\pi_x: T(X)_x \rightarrow T(X)_{\pi(x)} \cong G_1$ is surjective.*

Now we can prove the final theorem of this section.

THEOREM 2.9. *Let G_1 and G_2 be Δ -type minimal Δ -subgroups of \mathbb{G}_a with nonorthogonal generic types. Then there are proper Δ -subgroups G'_1 and G'_2 of G_1 and G_2 , respectively, such that G_1/G'_1 is isomorphic to G_2/G'_2 . In particular, G_1 and G_2 have the same Δ -type and typical Δ -dimension.*

PROOF. Since G_1 and G_2 are nonorthogonal, there is a small submodel K of \mathcal{U} and generic elements η_1, η_2 of G_1 and G_2 over K with $\eta_1 \not\perp_K \eta_2$. Let $X \subseteq G_1 \times G_2$ be the locus of (η_1, η_2) over K . Then X projects dominantly on both coordinates. Therefore, by passing to the tangent spaces at a suitable point, we obtain a proper subgroup H of $G_1 \times G_2$ which projects surjectively on G_1 and G_2 . Let

$$G'_1 = \{a \in G_1 : (a, 0) \in H\} \quad \text{and} \quad G'_2 = \{b \in G_2 : (0, b) \in H\}.$$

Clearly both G'_1 and G'_2 are proper subgroups. Moreover, the map sending $a + G'_1$ to $b + G'_2$ where $(a, b) \in H$ is a well defined isomorphism from G_1/G'_1 onto G_2/G'_2 . \dashv

§3. Interpreted fields. The main result in this section is the classification of fields interpreted in DCF_{0m} up to nonorthogonality. We begin with an analysis of subfields of \mathcal{U} . Our references are [18] and [7].

Let \mathbb{D} be the \mathcal{U} -space of all definable derivations on \mathcal{U} . Clearly \mathbb{D} has a natural Lie vector space structure. If \mathbb{D}' is a Lie subspace of \mathbb{D} , then the set

$$\mathcal{K}(\mathbb{D}') = \{a \in \mathcal{U} : \delta a = 0 \text{ for all } \delta \in \mathbb{D}'\}$$

is a subfield of \mathcal{U} . Moreover all definable subfields of \mathcal{U} are of this form.

THEOREM 3.1. (*Corollary 2.3.4 of [18]*) *The map sending \mathbb{D}' to $\mathcal{K}(\mathbb{D}')$ is an inclusion reversing one-to-one correspondence between Lie subspaces of \mathbb{D} and definable subfields of \mathcal{U} .*

This means that we need to understand the Lie subspaces of \mathbb{D} in order to understand the subfields of \mathcal{U} . The following theorem summarizes the basic results we need.

THEOREM 3.2. (*Propositions 6 and 7 in section 0.5 of [7]*) *A subspace of \mathbb{D} is a Lie subspace if and only if it has a commuting basis. Moreover any \mathcal{U} -independent commuting set of derivations can be extended to a basis of \mathbb{D} .*

This characterization has the following immediate corollary.

PROPOSITION 3.3. *Let \mathcal{M} be a definable subfield of \mathcal{U} . Then $(\mathcal{M}, +)$ is a Δ -type minimal subgroup of \mathbb{G}_a of typical Δ -dimension 1.*

PROOF. Let \mathbb{D}' be the Lie subspace of \mathbb{D} corresponding to \mathcal{M} . By 3.2, we can assume that $\{\delta_1, \dots, \delta_k\}$ is a basis of \mathbb{D}' for some k . Clearly $\{\delta_1, \dots, \delta_k\}$ is an autoreduced and coherent set, hence, by 2.6, it is a characteristic set for \mathcal{M} . So the dimension polynomial $\omega(s)$ of \mathcal{M} counts the elements of the set

$$\{(e_{k+1}, \dots, e_m) : e_{k+1} + \dots + e_m \leq s\}.$$

This means that $\omega(s) = \binom{s+\tau}{\tau}$ where $\tau = m - k$. In particular the typical Δ -dimension is 1. To prove Δ -type-minimality, we will use lemma 4.2.13 from [8] which says the following. Suppose

$$\omega(s) = \sum_{i=0}^{\tau} a_i \binom{s+i}{i}, \quad a_\tau \neq 0$$

is the dimension polynomial of a differential variety. If there is an $i < \tau$ such that $a_i \neq 0$ but $a_j = 0$ for $j = i + 1, \dots, \tau - 1$, then a_i is positive. Since a polynomial of degree τ which is lower than $\binom{s+\tau}{\tau}$ cannot have a positive coefficient other than its leading coefficient, it cannot be the dimension polynomial of a differential variety. \dashv

Next thing we will prove is that subfields of \mathcal{U} are essentially all the fields we have. To put it more precisely, we have the following theorem.

THEOREM 3.4. *Let \mathcal{M} be an infinite field interpreted in DCF_{0m} . Then there is a definable subfield \mathcal{M}' of \mathcal{U} which is definably isomorphic to \mathcal{M} .*

PROOF. The proof is a patchwork of citations. By elimination of imaginaries, we can assume that \mathcal{M} is definable. Hence the group $\text{PSL}(2, \mathcal{M})$ is a definable simple group. In corollary 4.2 of [11], it is proved that a definable group in DCF_{01} is isomorphic to a differential algebraic group.

Although it is not stated there, the same proof holds for partial differential fields. So we can assume that $\mathrm{PSL}(2, \mathcal{M})$ is a differential algebraic group. Thus it is isomorphic to a Chevalley group relative to some definable subfield \mathcal{M}' of \mathcal{U} by [5] and [13]. We know that the field structure on \mathcal{M}' is definable from the pure group structure on $\mathrm{PSL}(2, \mathcal{M}')$ and the multiplication on $\mathrm{PSL}(2, \mathcal{M})$, viewed as a Chevalley group, is definable from the pure field structure on \mathcal{M} . Therefore \mathcal{M}' is definably isomorphic to \mathcal{M} . \dashv

Since we know all the fields, the next step is to determine which subfields of \mathcal{U} are orthogonal. The answer to this question turns out to be quite easy. Any two distinct subfields of \mathcal{U} are orthogonal to each other. The orthogonality criterion we have tells us that we need to understand the quotients of the additive groups of fields. In other words, we need to know what happens to a field under an additive map.

THEOREM 3.5. *Let \mathcal{M} be a definable subfield of \mathcal{U} and let $\varphi: \mathcal{M} \rightarrow \mathbb{G}_a^n$ be a nontrivial definable additive map. Then there is another nontrivial definable additive map φ' from $\varphi(\mathcal{M})$ onto \mathcal{M} .*

PROOF. Note that it suffices to prove the proposition for $n = 1$. For notational simplicity we will assume that the field is defined by the system

$$\delta_1 y = 0, \delta_2 y = 0, \dots, \delta_k y = 0.$$

Let τ be the Δ -type of \mathcal{M} . Write $\varphi(y) = \sum a_\theta \theta y$. Let K be a small submodel of \mathcal{U} over which \mathcal{M} and φ are defined. Fix a generic element η of \mathcal{M} over K . Now let s be the order of φ and let V be the K -space generated by the set $\Theta(s)y$. Clearly, if δ is a derivation on K then the polynomial $\varphi^{\delta^i}(y) := \sum \delta^i(a_\theta)\theta y$ is in V for each $i \geq 0$. Therefore, as V is finite dimensional, there are a $d > 0$ and $a_0, \dots, a_d \in K$ with $a_d \neq 0$ such that $\sum_{i=0}^d a_i \varphi^{\delta^i}(y) = 0$. Hence, if we choose δ to be one of $\delta_1, \dots, \delta_k$, we get

$$\sum_{i=0}^d a_i \delta^i(\varphi(\eta)) = \sum_{i=0}^d \sum_{j=0}^i a_i \binom{i}{j} \varphi^{\delta^j}(\delta^{i-j}\eta) = \sum_{i=0}^d a_i \varphi^{\delta^i}(\eta) = 0.$$

This means that for each $i = 1, \dots, k$ there is a nonzero linear homogeneous δ_i -polynomial f_i with coefficients in K such that $\varphi(\mathcal{M}) \subseteq \ker(f_i)$. Let $G = \bigcap_{i=1}^k \ker(f_i)$. Obviously G is a vector space over \mathcal{M} . By looking at the leaders of f_i , we can see that the Δ -type of G is at most τ . But G contains $\varphi(\mathcal{M})$, which has Δ -type exactly τ as \mathcal{M} is Δ -type minimal. Thus the Δ -type of G is τ as well. In particular, G is a finite dimensional vector space over \mathcal{M} , say of dimension r . So, after fixing a basis of G , we get r many projection maps from G onto \mathcal{M} . Denote these maps by π_1, \dots, π_r . Note that for each i the restriction of π_i to $\varphi(\mathcal{M})$, denoted

by $\pi_i|_{\varphi(\mathcal{M})}$, is either surjective or zero by Δ -type minimality. Since they cannot be all trivial, there is an i such that $\pi_i|_{\varphi(\mathcal{M})}$ is surjective. \dashv

Now we are ready to prove the final theorem of this section.

THEOREM 3.6. *Let \mathcal{M}_1 and \mathcal{M}_2 be distinct definable subfields of \mathcal{U} . Then the generic types of \mathcal{M}_1 and \mathcal{M}_2 are orthogonal.*

PROOF. Suppose they are not orthogonal. Then there are additive proper subgroups $A_1 \subseteq \mathcal{M}_1$ and $A_2 \subseteq \mathcal{M}_2$ such that \mathcal{M}_1/A_1 is isomorphic to \mathcal{M}_2/A_2 . The previous theorem says there is a homomorphism from \mathcal{M}_2/A_2 onto \mathcal{M}_2 . By composing it with the isomorphism between \mathcal{M}_2/A_2 and \mathcal{M}_1/A_1 , we obtain a homomorphism from \mathcal{M}_1/A_1 to \mathcal{M}_2 . Clearly this map is surjective as \mathcal{M}_1/A_1 and \mathcal{M}_2 are of the same Δ -type. Thus we proved that \mathcal{M}_2 is a homomorphic image of \mathcal{M}_1 .

Again we can assume that \mathcal{M}_1 is defined by

$$\delta_1 y = 0, \delta_2 y = 0, \dots, \delta_k y = 0.$$

Let φ be a homomorphism from \mathcal{M}_1 onto \mathcal{M}_2 . Clearly there is a finite subset Θ^* of the the free commutative monoid generated by $\delta_{k+1}, \dots, \delta_m$ such that

$$\varphi(y) = \sum_{\theta \in \Theta^*} a_\theta \theta y.$$

Choose a definable derivation δ that vanishes on \mathcal{M}_2 but not on \mathcal{M}_1 . Such a derivation exists because \mathcal{M}_1 is not contained in \mathcal{M}_2 by Δ -type minimality. Let $\alpha_1, \dots, \alpha_m$ in \mathcal{U} be such that $\delta = \alpha_1 \delta_1 + \dots + \alpha_m \delta_m$. As δ does not vanish on \mathcal{M}_1 there is an $i \in \{k+1, \dots, m\}$ such that $\alpha_i \neq 0$. Let K be a small submodel of \mathcal{U} over which $\mathcal{M}_1, \mathcal{M}_2, \delta$ and φ are defined. Choose a generic element η of \mathcal{M}_1 over K . Then

$$0 = \delta(\varphi(\eta)) = \varphi^\delta(\eta) + \varphi(\delta\eta) = \varphi^\delta(\eta) + \varphi(\alpha_{k+1}\delta_{k+1}\eta + \dots + \alpha_m\delta_m\eta).$$

Let θy be the leader of $\varphi(y)$ with respect to some fixed orderly ranking. Then the leader of

$$\varphi^*(y) := \varphi^\delta(y) + \varphi(\alpha_{k+1}\delta_{k+1}y + \dots + \alpha_m\delta_my)$$

is $\delta_j \theta$ for some $j \in \{k+1, \dots, m\}$. Note that $\varphi^*(y)$ is in the defining ideal of \mathcal{M}_1 over K but its leader is not a derivative of any of $\delta_1 y, \delta_2 y, \dots, \delta_k y$. This is a contradiction since $\{\delta_1 y, \delta_2 y, \dots, \delta_k y\}$ is a characteristic set of \mathcal{M}_1 . \dashv

§4. Rank Inequalities. There are several notions of rank for types in differentially closed fields other than the Morley rank and the Lascar rank. Here, we will introduce only one of them. Let us fix a small submodel K of \mathcal{U} and let p be a type over K . We know that p is determined by a prime ideal \mathfrak{p} of $K\{y\}$. The rank we will define is the differential analog of Krull dimension.

DEFINITION 4.1. Let $\Delta\text{-Spec}(K\{y\})$ be the collection of all prime Δ -ideals of $K\{y\}$. Define the Δ -dimension of p , denoted by $\Delta\text{-dim}(p)$, as follows:

$$\Delta\text{-dim}(p) = \sup\{\Delta\text{-dim}(\mathfrak{p}') + 1 : \mathfrak{p} \in \Delta\text{-Spec}(K\{y\}), \mathfrak{p}' \supset \mathfrak{p}\}.$$

This definition makes sense since $\Delta\text{-Spec}(K\{y\})$ has the ascending chain property.

The first result about ranks in differentially closed fields was proved by Pong in [14] for the case of DCF_{01} .

THEOREM 4.2. *Suppose \mathcal{U} is an ordinary differentially closed field and let p be a type over K with dimension polynomial $\omega_p(s) = as + b$. Then*

$$\omega a \leq \text{RU}(p) \leq \text{RM}(p) \leq \Delta\text{-dim}(p) < \omega(a + 1).$$

McGrail, in [8], claimed to prove the following generalization to the case of several derivations.

Let p be a type over K with Δ -type τ and typical Δ -dimension a_τ . Then

$$\omega^\tau a_\tau \leq \text{RU}(p) \leq \text{RM}(p) \leq \Delta\text{-dim}(p) < \omega^\tau(a_\tau + 1).$$

Unfortunately, this is not completely correct. First of all, McGrail's lower bound does not specialize to Pong's lower bound for the case of a single derivation. If p has finite transcendence degree, that is, if $\tau = 0$, then Pong's inequalities only say that $0 \leq \text{RU}(p) < a_\tau + 1$. However, McGrail's lower bound implies that $a_\tau \leq \text{RU}(p) < a_\tau + 1$, which means the Lascar rank of p and the typical Δ -dimension of p are equal. Clearly, the generic type of the Manin kernel or the generic type of the equation $\delta y = y\delta^2 y$ are counterexamples to this claim as they are both of Lascar rank 1 while their typical Δ -dimensions are greater than 1.

This was realized by Pong, who used McGrail's inequalities only in the case of positive Δ -type. See, for instance, [12] and [15]. However, the lower bound is not valid for the case of positive Δ -type either. For the moment, the most general theorem we have is the following, whose proof can be found in [15].

THEOREM 4.3. *Let p be a type over K with Δ -type τ and typical Δ -dimension a_τ . Then*

$$\text{RU}(p) \leq \text{RM}(p) \leq \Delta\text{-dim}(p) < \omega^\tau(a_\tau + 1).$$

It turns out that no nontrivial lower bound can depend on the typical Δ -dimension of the type. In other words, it is possible to fix the Lascar rank while increasing the typical Δ -dimension arbitrarily. We will construct a family of types illustrating this fact. The first result we need is an easy characterization of subgroups of \mathbb{G}_a which have Lascar rank ω . In the rest of this section we will work in DCF_{02} .

PROPOSITION 4.4. *Let G be a definable subgroup of \mathbb{G}_a with generic type p . Then $\text{RU}(p) = \omega$ if and only if G has infinite transcendence degree but every proper definable subgroup of G has finite transcendence degree.*

PROOF. Let G be a group of Lascar rank ω . Clearly it has infinite transcendence degree as $\text{RU}(p) \leq \Delta\text{-dim}(p)$. Moreover, if H is a proper subgroup of G , then it is of finite Lascar Rank. Hence H is a finite dimensional vector space over the constants. In particular, its transcendence degree is finite.

Now suppose $G \subseteq \mathbb{G}_a$ is a group of infinite transcendence degree but any proper subgroup of G is of finite transcendence degree. Clearly G is not \mathbb{G}_a . So the Lascar rank of the generic type of G , $\text{RU}(p)$, is less than ω^2 . Hence there are a, b in ω such that $\text{RU}(p) = \omega a + b$. As G is an infinite dimensional vector space over the constants we have $a \geq 1$. Note that it suffices to prove $\text{RM}(G) \leq \omega$ as $\text{RU}(p) \leq \text{RM}(G)$. Suppose $\text{RM}(G) > \omega$. Then G has a proper definable subset X of Morley rank ω . We can assume that X is closed. Now for a smooth point α of X , the differential tangent space $T(X)_\alpha$ is a proper subgroup of $T(G)_\alpha \cong G$. But X and $T(X)_\alpha$ have the same dimension polynomial and X is of infinite transcendence degree. Therefore $T(G)_\alpha$ gives a proper subgroup of G whose transcendence degree is infinite. Contradiction. \dashv

Now we can give our first example.

PROPOSITION 4.5. *Let $f \in \mathcal{U}[\delta_2]$ be of order $k > 0$ and let p be the generic type of the group defined by the equation $\delta_1 y = f(y)$. Then*

1. p has Δ -type 1 and typical Δ -dimension k ,
2. $\text{RU}(p) = \omega$.

PROOF. Since $\delta_1 y - f(y)$ is linear, 2.6 implies that the singleton consisting of $\delta_1 y - f(y)$ is a characteristic set of the Δ -ideal it generates. Hence the dimension polynomial of p is

$$\omega_p(s) = k(s+1) - \frac{k(k-1)}{2}.$$

This proves the first part. For the second part, we will use proposition 4.4. Let H be a proper subgroup of G defined over some small differentially closed field K which contains the coefficients of f . Since H is strictly contained in G , there is a homogeneous linear polynomial $g(y)$ which vanishes on H but not on G . Let us fix a ranking of $K\{y\}$ in which $\delta_1 y$ is higher than $\delta_2^n y$ for all $n \geq 0$. Now by reducing $g(y)$ with respect to $\delta_1 y - f(y)$ we can assume that

$$g(y) = \sum_{i=0}^{d+1} a_i \delta_2^i y$$

for some $d \geq 0$ and a_1, \dots, a_d in K . Let η be a generic element of H over K . Since $\delta_1\eta = f(\eta)$ and the coefficients of f are in K , we have

$$K\langle\eta\rangle = K(\delta_2^j\eta: j = 0, 1, 2, \dots).$$

On the other hand $g(\eta) = 0$. Hence

$$K(\delta_2^j\eta: j = 0, 1, 2, \dots) = K(\delta_2^j\eta: j = 0, 1, 2, \dots, d).$$

Therefore the transcendence degree of $K\langle\eta\rangle$ over K is at most $d + 1$. \dashv

The idea here was to replace each occurrence of δ_1 by δ_2 . As we used a nonorderly ranking to do that, the dimension polynomial could not detect the trick. The following example is of a different sort and has a more combinatorial feeling. Even though the idea is applicable to a wide range of groups, we will work with a specific one for the sake of simplicity.

PROPOSITION 4.6. *Let p be the generic type of the group defined by the equation $\delta_1^2y = \delta_2^3y$. Then*

1. p has Δ -type 1 and typical Δ -dimension 3,
2. $\text{RU}(p) = \omega$.

PROOF. The first part follows from the fact that $\{\delta_1^2y - \delta_2^3y\}$ is a characteristic set. Let H be a proper subgroup of G defined over a small differentially closed field K . This time we will fix an orderly ranking in which δ_1y is higher than δ_2y . As in the previous example, there is a linear homogeneous polynomial $f(y)$ which vanishes on H but not on G . We can assume that $f(y)$ is reduced with respect to $\delta_1^2y - \delta_2^3y$. Such a polynomial is of the form

$$f(y) = f_0(y) + f_1(\delta_2y) + f_2(\delta_2^2y)$$

where f_0, f_1, f_2 are in $K[\delta_1]$. If we apply δ_2 to $f(y)$ and replace δ_2^3y by δ_1^2y we obtain another polynomial which vanishes on H , namely

$$f^*(y) = (f_0^{\delta_2} + f_2 \circ \delta_1^2)(y) + (f_1^{\delta_2} + f_0)(\delta_2y) + (f_2^{\delta_2} + f_1)(\delta_2^2y).$$

Note that it is enough to find a polynomial vanishing on H whose leader is of the form $\delta_1^n y$ for some n because this implies that the transcendence degree of H is at most $3n$. So, in particular, if the leader of $f(y)$ comes from $f_0(y)$ we are done. If the leader of $f(y)$ comes from $f_2(\delta_2^2y)$, then the $f^*(y)$ does the job. Finally, if the leader of $f(y)$ comes from $f_1(\delta_2y)$, we apply the same procedure twice. So, in any case, we can find a polynomial of the desired form. \dashv

§5. Groups of Δ -type 1. Common wisdom tells us that the study of ordinary differential fields is strictly easier than the study of partial differential fields. One of the reasons of this phenomenon is the following classical theorem whose proof can be found in [19].

THEOREM 5.1. *Let K be a field and let δ be a derivation on K . Then $K[\delta]$, the ring of linear δ -operators with coefficients in K , is a left and right principal ideal domain.*

This gives rise to a nice structure theory for finitely generated $K[\delta]$ -modules which has no analogue for $K[\Delta]$ -modules in general.

From now on we will only work with left modules.

THEOREM 5.2. *Let M be a finitely generated $K[\delta]$ -module. Then $M = F \oplus T$ where F is a free $K[\delta]$ -module and T is torsion. Moreover, the rank of the free part does not depend on the decomposition.*

In the case of ordinary differential fields these are enough to classify all the subgroups of \mathbb{G}_a^n up to isomorphism. See [4] for details.

THEOREM 5.3. *Suppose \mathbb{G}_a is the additive group of an ordinary differentially closed field. Let G be a definable subgroup of \mathbb{G}_a^n . Then there are natural numbers k_1 and k_2 such that G is definably isomorphic to $\mathbb{G}_a^{k_1} \oplus \mathcal{K}^{k_2}$ where \mathcal{K} is the field of constants.*

We know that there is no such simple classification in the case of partial differential fields. For instance, the group defined by the equation $\delta_1 y = \delta_2^k y$ is not isomorphic to a product of additive groups of fields if k is greater than 1. However, the theory of $K[\delta]$ -modules is not totally useless for the case of several derivations since we do not need all the derivations most of the time. For definable groups G and H , let $\text{Hom}(G, H)$ be the set of all definable group homomorphisms from G to H .

THEOREM 5.4. *Let $G \subseteq \mathbb{G}_a$ be a definable subgroup of Δ -type τ . Then there is a set Δ' of commuting definable derivations with $|\Delta'| = \tau$ such that $\text{Hom}(G, \mathbb{G}_a)$ is a finitely generated $\mathcal{U}[\Delta']$ -module.*

PROOF. Let G be a group of Δ -type τ with defining ideal $\mathfrak{p} \subseteq \mathcal{U}\{y\}$. By lemma 2.3 of [17], there are definable commuting derivations $\delta'_1, \dots, \delta'_m$ and an integer t such that for each i in $\{\tau + 1, \dots, m\}$, there is a linear homogeneous polynomial in \mathfrak{p} whose leader is $\delta_i^t y$. Let Θ' be the free commutative monoid generated by $\delta'_1, \dots, \delta'_m$. Fix a linear characteristic set Λ of \mathfrak{p} . Recall if $\theta \in \Theta'$, θy is the leader of a linear homogeneous polynomial in \mathfrak{p} if and only if there is an element in Λ with leader $\theta' y$ such that $\theta' | \theta$. Hence, if Θ'_0 is a finite subset of Θ' and

$$f(y) = \sum_{\theta \in \Theta'_0} a_\theta \theta y, \quad a_\theta \neq 0 \text{ for } \theta \in \Theta'_0$$

is reduced with respect to Λ , then δ_i^t does not divide θ for $\theta \in \Theta'_0$ and i in $\{\tau + 1, \dots, m\}$. In particular, f is in the \mathcal{U} -space generated by

$$\{\delta_1^{e_1} \cdots \delta_m^{e_m} : e_{\tau+1}, \dots, e_m < t\}.$$

But this vector space is also the $\mathcal{U}[\Delta'_\tau]$ -module generated by

$$\{\delta'_{\tau+1}{}^{e_{\tau+1}} \cdots \delta'_m{}^{e_m} : e_{\tau+1}, \dots, e_m < t\}$$

where $\Delta'_\tau = \{\delta'_1, \dots, \delta'_\tau\}$. This proves that $\text{Hom}(G, \mathbb{G}_a)$ is a finitely generated $\mathcal{U}[\Delta'_\tau]$ -module since any element of $\text{Hom}(G, \mathbb{G}_a)$ can be expressed by a linear homogeneous Δ -polynomial which is reduced with respect to Λ . \dashv

This implies that if G is a group of Δ -type 1, then $\text{Hom}(G, \mathbb{G}_a)$ is a $K[\delta]$ -module for some definable derivation δ . Note that for different derivations we get different modules with different invariants. In particular the rank of the free part may change. For example, let G be the group defined by $\delta_1 y = \delta_2^2 y$. Then $\text{Hom}(G, \mathbb{G}_a)$ is a free $\mathcal{U}[\delta_2]$ -module of rank 1 and a free $\mathcal{U}[\delta_1]$ -module of rank 2. But these are the only possibilities because of the next theorem.

THEOREM 5.5. *Let G be a Δ -type minimal subgroup of \mathbb{G}_a with Δ -type 1. Suppose that $\text{Hom}(G, \mathbb{G}_a)$ is a finitely generated $\mathcal{U}[\delta]$ -module (for some definable δ) with $\mathcal{U}[\delta]$ -rank e . Then e is less than or equal to the typical Δ -dimension of G .*

PROOF. Let $\varphi_1, \dots, \varphi_e$ be $\mathcal{U}[\delta]$ -independent elements of $\text{Hom}(G, \mathbb{G}_a)$. Define $\varphi: G \rightarrow \mathbb{G}_a^e$ by

$$y \mapsto (\varphi_1(y), \dots, \varphi_e(y)).$$

Since G is Δ -type minimal, G and $\varphi(G)$ have the same typical Δ -dimension. Moreover if η is a generic element of G over \mathcal{U} (from an extension of \mathcal{U}) then $\varphi(\eta)$ is a generic element of $\varphi(G)$ over \mathcal{U} . So it is enough to prove that

$$\text{tr-deg}_{\mathcal{U}} \mathcal{U}(\delta^j \varphi_i(\eta) : j = 0, \dots, s; i = 1, \dots, e) \geq es.$$

Our assumption says that the set

$$\{\delta^j \varphi_i(\eta) : j = 0, \dots, s; i = 1, \dots, e\}$$

is linearly independent over \mathcal{U} , and this is enough to prove our theorem. See the second section of [18] for details. \dashv

Now we can classify all Δ -type 1 subgroups of \mathbb{G}_a which are nonorthogonal to fields.

THEOREM 5.6. *Let G be a Δ -type minimal subgroup of \mathbb{G}_a with Δ -type 1. Then G is nonorthogonal to a field if and only if its typical Δ -dimension is 1.*

PROOF. We know that if G is nonorthogonal to a field then its typical Δ -dimension has to be 1. For the other direction we will do induction on $|\Delta| = m$ and prove the following slightly stronger statement. If $G \subseteq \mathbb{G}_a$ is

a Δ -type minimal subgroup of typical Δ -dimension 1 which is nonorthogonal to a field, then there is a definable additive map from G onto a definable subfield of \mathcal{U} .

If $m = 1$ then G is the whole field. So we can assume that $m > 1$ and the statement holds for all integers less than m . Let $M = \text{Hom}(G, \mathbb{G}_a)$. As G is of Δ -type 1 we know that M is a finitely generated module over $\mathcal{U}[\delta]$ for some definable derivation δ . We will assume that $\delta = \delta_1$ for the sake of notational simplicity. As G is infinite dimensional over the constants, M is not torsion and hence the rank of the free part of M is not zero. On the other hand, by 5.5, the rank of the free part of M is at most 1. Thus there is a ψ in M such that

$$M = \mathcal{U}[\delta_1]\psi \oplus T$$

where T is the torsion part of M . We will consider two cases.

Case 1. T is nontrivial. Take a nonzero element φ in T . Then there is a nonzero f in $\mathcal{U}[\delta_1]$ such that $f\varphi = 0$. So $\varphi(G) \subseteq \ker(f)$. But $\ker(f)$ is a finite dimensional vector space over $\ker(\delta_1)$. Hence, after fixing a basis, we can construct a nonzero homomorphism from $\varphi(G)$ to $\ker(\delta_1)$ by projection. Let \tilde{G} be such a homomorphic image of G in $\ker(\delta_1)$. Note that $\ker(\delta_1)$, viewed as a $\{\delta_2, \dots, \delta_m\}$ -field, is a model of $\text{DCF}_{0, m-1}$. Moreover \tilde{G} satisfies the induction hypothesis relativized to $\ker(\delta_1)$. So there is a surjective group homomorphism from \tilde{G} to a Δ -type 1 field living in $\ker(\delta_1)$. Therefore there is a homomorphism from G onto a Δ -type 1 field living in \mathcal{U} .

Case 2. M is torsion free, that is $M = \mathcal{U}[\delta_1]\psi$. We will denote $\psi(G)$ by \tilde{G} . Let φ be an element of $\text{Hom}(\tilde{G}, \mathbb{G}_a)$. Clearly $\varphi \circ \psi$ gives an element of $\text{Hom}(G, \mathbb{G}_a)$ which has to be of the form $f\psi$ for some f in $\mathcal{U}[\delta_1]$. Thus $\varphi \circ \psi = f\psi$. So φ and f have the same action on \tilde{G} . We just proved that any element of $\text{Hom}(\tilde{G}, \mathbb{G}_a)$ can be represented by a linear homogeneous polynomial in $\mathcal{U}[\delta_1]$. Since such a representation has to be unique, we can identify $\text{Hom}(\tilde{G}, \mathbb{G}_a)$ with $\mathcal{U}[\delta_1]$. In particular, the action of $\delta_2, \dots, \delta_m$ on \tilde{G} can be expressed in terms of δ_1 , i.e. there are f_2, \dots, f_m in $\mathcal{U}[\delta_1]$ such that the polynomials $\delta_i y - f_i(y)$ are in the defining ideal of \tilde{G} for $i = 2, \dots, m$. We will show that each f_i is of order 1.

Let η be a generic element of \tilde{G} over \mathcal{U} . We will fix an i in $\{2, \dots, m\}$ and denote the set $\{\delta_1, \delta_i\}$ by Δ_i . Let \tilde{G}_i be the group defined by $\delta_i y = f_i(y)$. Note that if we view \tilde{G}_i as a Δ_i -group, then \tilde{G} is a dense (and not definable) subset of \tilde{G}_i . Indeed, the Δ_i -closure of \tilde{G} is a subgroup of \tilde{G}_i and any proper Δ_i -subgroup of \tilde{G}_i is contained in the kernel of an element of $\mathcal{U}[\delta_1]$ by the proof of 4.5. Since η , an element of \tilde{G} , cannot satisfy this, \tilde{G} is dense in \tilde{G}_i . Therefore η is also a generic element of \tilde{G}_i viewed as a

Δ_i -group. Obviously

$$\text{tr-deg}_{\mathcal{U}}\mathcal{U}(\theta\eta: \theta \in \Theta_i(s)) \leq \text{tr-deg}_{\mathcal{U}}\mathcal{U}(\theta\eta: \theta \in \Theta(s))$$

where $\Theta_i(s)$ is the intersection of $\Theta(s)$ with the commutative monoid generated by Δ_i . This forces the typical Δ_i -dimension of \tilde{G}_i , which is also the order of f_i , to be 1. So there are a_i, b_i in \mathcal{U} for $i = 2, \dots, m$ such that

$$\tilde{G} \subseteq \bigcap_{i=2}^m \ker(\delta_i + a_i\delta_1 + b_i).$$

The right hand side is a one dimensional vector space over the field given by $\mathcal{M} = \bigcap_{i=2}^m \ker(\delta_i + a_i\delta_1)$. But then \tilde{G} is equal to that vector space by Δ -type minimality and hence there is a surjection from \tilde{G} onto \mathcal{M} . This finishes the proof. \dashv

§6. Some generalities. First, we will derive a necessary condition for a class of subgroups of \mathbb{G}_a to be locally modular. Let us fix a $G \subseteq \mathbb{G}_a$ defined over a small elementary submodel K of \mathcal{U} . Suppose $\tau(G) = 1$, $\text{RU}(G) = \omega$ and the generic type of $p \in S_1(K)$ of G is non-locally modular. By non-local modularity, there is a two dimensional family of curves living in $G \times G$. We will show that one can choose a family of linear curves with this property. Our strategy is to pass to the tangent space at a suitable point and to obtain a p -semi-regular subgroup of $G \times G$ whose canonical parameter has positive p -weight.

It is worth mentioning that without the p -semi-regularity requirement it is very easy to construct such a group. Let α be a generic element of G . Define

$$H_\alpha = \{(a, a + c\alpha) : a \in G, c \text{ is a constant}\}.$$

Then the generic type of H_α is p -simple of p -weight 1 and the canonical parameter of H_α is the logarithmic derivative of α , which has positive p -weight. Note that $H_\alpha = H'_\alpha \oplus H''_\alpha$ where H'_α is the diagonal subgroup of $G \times G$ and H''_α is the vector space over the constants generated by $(0, \alpha)$. The following proposition tells us that this is not a coincidence. It also gives us a characterization of p -semi-regular subgroups of $G \times G$.

PROPOSITION 6.1. *Let $H \subseteq \mathbb{G}_a$ be a definable subgroup of Δ -type 1 with $\text{RU}(H) < \omega 2$. Then there are definable subgroups H' and H'' of H such that $H = H' \oplus H''$, $\text{RU}(H') = \omega$ and $\text{RU}(H'') < \omega$. In particular the following are equivalent:*

1. *The generic type of H is regular.*
2. *$\text{RU}(H) = \omega$.*
3. *H is Δ -type minimal.*

PROOF. First note that any subgroup of \mathbb{G}_a with Δ -type greater than 0 has infinite Lascar rank because any such group is an infinite dimensional vector space over the constants. Thus $\text{RU}(H) = \omega + d$ for some integer d . This implies that H has a definable subgroup H' of Lascar rank ω . Note that both H and H' are vector spaces over the constants. Hence H' has a d dimensional complement H'' as an abstract vector space. But any finite dimensional vector space over the constants is definable. So H decomposes into a direct sum in a definable way.

These arguments also prove that $\text{RU}(H) = \omega$ if and only if H is Δ -type minimal. Since a type of rank ω is always regular, we only need to show that if the generic type of H is regular then it is Δ -type minimal. Suppose the generic type of H , say q , is regular but H is not Δ -type minimal. Clearly q is nonorthogonal to the generic types of H' and H'' because of the projection maps. Since q is regular, this implies that the generics of H' and H'' are nonorthogonal. But this is impossible as $\text{RU}(H') = \omega$ and $\text{RU}(H'') < \omega$. \dashv

Before deriving our condition we need to address another issue. If q is a regular type and X is its locus, then it is not always true that $T(X)_\alpha$ is a regular group even if α is a smooth point. A simple example is the generic type of $\delta y = y\delta^2 y$ in the case of ordinary differential fields. The tangent space of the locus of this type at a smooth point has the same dimension polynomial as the locus, which implies that it is a 2 dimensional vector space. In particular, it is not regular. Now we will prove that this does not happen in our case.

PROPOSITION 6.2. *Let $(\eta_1, \eta_2) \in G \times G$ be such that $\text{tp}(\eta_1\eta_2/K)$ is p -semi-regular of p -weight 1 and let $X = \text{loc}(\eta_1\eta_2/K)$. Then $H := T(X)_{(\eta_1, \eta_2)}$ is p -semi-regular of p -weight 1.*

PROOF. We will view H as a subgroup of $G \times G$ by using the fact that $T(G \times G)_{(\eta_1, \eta_2)}$ is isomorphic to $G \times G$. Clearly $\text{RU}(H) < \text{RU}(G \times G) = \omega 2$. So it is enough to show that $\text{RU}(H) = \omega$ by proposition 6.1. Suppose $\text{RU}(H) > \omega$. Then H decomposes into a nontrivial direct sum. In particular, there is a group homomorphism φ' from H to \mathbb{G}_a^n such that $\varphi'(H)$ is finite dimensional. We know that φ extends to a morphism from $G \times G$ to \mathbb{G}_a^n which, in turn, gives a map from X to \mathbb{G}_a^n as $X \subseteq G \times G$. We will denote this map by φ . So φ and φ' are defined by the same linear differential polynomial but their domains are different. If we differentiate φ at (η_1, η_2) we do not change the polynomial defining φ as it is linear, however the domain becomes H . Therefore the derivative of φ at (η_1, η_2) is precisely φ' , whose image is finite dimensional. This implies that the closure of $\varphi(X)$ is of finite transcendence degree. But then $\text{tp}(\eta_1\eta_2/K)$ is nonorthogonal to a type of finite rank and this violates p -semi-regularity as p is of rank ω . \dashv

Here is the result we promised. In the proof, which is based on an argument by Anand Pillay, we make use of the theory of differential arc spaces developed in [9]. For an irreducible Δ -closed set X and a natural number i we denote the i -th arc bundle of X by $\mathcal{A}_i(X)$. If η is an element of X , then $\mathcal{A}_i(X)_\eta$ is the i -th arc space of X at η .

THEOREM 6.3. *There is a p -semi-regular subgroup $H \subseteq G \times G$ of p -weight 1 whose canonical parameter has positive p -weight.*

PROOF. by [10] there are an $L \supseteq K$, a pair $\eta = (\eta_1, \eta_2)$ and a tuple c such that

- η_1 and η_2 are independent realizations of $p|L$,
- $\text{stp}(\eta/Lc)$ is p -semi-regular of p -weight 1,
- $\text{stp}(c/L)$ is p -semi-regular of p -weight 2 and
- $c = \text{Cb}(\text{stp}(\eta/Lc))$.

Let $X = \text{loc}(\eta/Lc)$. By 6.2, we know that $T(X)_\eta$, which is equal to $\mathcal{A}_1(X)_\eta$ by 2.1 of [9], is regular. So it suffices to show that the canonical parameter of $\mathcal{A}_1(X)_\eta$ has positive p -weight. For $i < \omega$ let a_i be the canonical parameter of $\mathcal{A}_i(X)_\eta$ viewed as a subset of $\mathcal{A}_i(G \times G)_\eta$. By 2.6 of [9], there is a natural number k such that $a := (a_1, \dots, a_k)$ is interdefinable with c over $\text{dcl}(L\eta)$. By expanding $w_p(c\eta/L)$ in two different ways we get

$$w_p(c/L) + w_p(\eta/Lc) = w_p(\eta/L) + w_p(c/L\eta).$$

Thus $1 = w_p(c/L\eta) = w_p(a/L\eta)$. So there is a minimum i with $w_p(a_i/L\eta) = 1$. Note that if we can show $i = 1$ then we are done since $K \subseteq \text{dcl}(L\eta)$ and hence $w_p(a_1) \geq w_p(a_1/L\eta)$. Suppose $i > 1$. For the sake of notational simplicity we name L , η and a_1 . By 2.1 of [9] again, there is a natural projection map

$$\pi: \mathcal{A}_i(X)_\eta \rightarrow \mathcal{A}_{i-1}(X)_\eta$$

such that $\pi^{-1}(\alpha)$ is isomorphic to $\mathcal{A}_1(X)_\eta$ for each α in $\mathcal{A}_{i-1}(X)_\eta$. So, by the choice of i , $\pi^{-1}(\alpha)$ is defined over $\text{cl}_p(\alpha)$. Let $(\alpha_j: j < \omega)$ be a tuple of elements of $\mathcal{A}_{i-1}(X)_\eta$ which are generic and independent over a_i . We will show that a_i is in the definable closure of $(\text{cl}_p(\alpha_j): j < \omega)$. Let σ be an automorphism of \mathcal{U} fixing each $\text{cl}_p(\alpha_j)$. There is a j' such that $\alpha_{j'}$ is independent from $a_i\sigma(a_i)$. Choose a generic element ξ of $\pi^{-1}(\alpha_{j'})$ over $a_i\sigma(a_i)$. Note that ξ is not only a generic element of $\pi^{-1}(\alpha_{j'})$ but also a generic element of $\mathcal{A}_i(X)_\eta$ and $\sigma(\mathcal{A}_i(X)_\eta)$ over $a_i\sigma(a_i)$. Therefore σ fixes $\mathcal{A}_i(X)_\eta$ setwise and hence a_i is in the definable closure of $(\text{cl}_p(\alpha_j): j < \omega)$. But $w_p(a_i) = w_p(a_i/\alpha_j: j < \omega)$ as $a_i \perp \{\alpha_j: j < \omega\}$. Contradiction. \dashv

In 4.5 we introduced a class of regular groups. Now we will investigate those groups in detail. As in 4.5 we will work in DCF_{02} . For an element f in $\mathcal{U}[\delta_2]$, let E_f be the subgroup of \mathbb{G}_a defined by $\delta_1 y = f(y)$. Here are

a few basic properties of such groups. Some of the arguments should be familiar as they were used in other proofs.

PROPOSITION 6.4. *Let $E \subseteq \mathbb{G}_a$ be a definable subgroup.*

1. $E = E_f$ for some f in $\mathcal{U}[\delta_2]$ if and only if $\text{Hom}(E, \mathbb{G}_a) = \mathcal{U}[\delta_2]$.
2. E is definably isomorphic to E_f for some f in $\mathcal{U}[\delta_2]$ if and only if $\text{Hom}(E, \mathbb{G}_a)$ is a free $\mathcal{U}[\delta_2]$ -module of rank 1.
3. If $E = E_f$ for some f in $\mathcal{U}[\delta_2]$, then for any proper subgroup E' of E there is an f' in $\mathcal{U}[\delta_2]$ such that E/E' is definably isomorphic to $E_{f'}$.

PROOF. 1. From left to right is clear. Suppose $\text{Hom}(E, \mathbb{G}_a) = \mathcal{U}[\delta_2]$. Then the homomorphism given by δ_1 can be expressed by an element $f \in \mathcal{U}[\delta_2]$. So $\delta_1(y) = f(y)$ vanishes on E , which means that $E \subseteq E_f$. All proper subgroups of E_f are finite dimensional by 4.5. But E is not finite dimensional since $\text{Hom}(E, \mathbb{G}_a)$ is not torsion. Hence $E = E_f$.

2. Suppose there is a φ in $\text{Hom}(E, E_f)$ which is an isomorphism. Then the map from $\text{Hom}(E_f, \mathbb{G}_a)$ to $\text{Hom}(E, \mathbb{G}_a) = \mathcal{U}[\delta_2]$ given by $\psi \mapsto \psi \circ \varphi$ is a module isomorphism. For the other direction, suppose $\text{Hom}(E, \mathbb{G}_a) = \mathcal{U}[\delta_2]\varphi$ for some φ in $\text{Hom}(E, \mathbb{G}_a)$. As the inclusion map ι is also a homomorphism from E to \mathbb{G}_a , there is a g in $\mathcal{U}[\delta_2]$ such that $\iota = g\varphi$. Now if α is in $\ker(\varphi)$ then

$$\alpha = \iota(\alpha) = g\varphi(\alpha) = g(0) = 0.$$

Therefore φ is injective. Let \tilde{E} be $\varphi(E)$. Clearly E is isomorphic to \tilde{E} . So we are done if we show that $\text{Hom}(\tilde{E}, \mathbb{G}_a) = \mathcal{U}[\delta_2]$. Let ψ be in $\text{Hom}(\tilde{E}, \mathbb{G}_a)$. Note that $\psi \circ \varphi$ is in $\text{Hom}(E, \mathbb{G}_a)$. So, by assumption, there is a g in $\mathcal{U}[\delta_2]$ such that $g\varphi = \psi \circ \varphi$. Therefore g and ψ have the same action on \tilde{E} .

3. Let

$$M = \{\psi \in \text{Hom}(E, \mathbb{G}_a) : E' \subseteq \ker(\psi)\}.$$

Then M is a nonzero submodule of $\text{Hom}(E, \mathbb{G}_a) = \mathcal{U}[\delta_2]$. So there is a φ in M such that $M = \mathcal{U}[\delta_2]\varphi$. Clearly $E' = \ker(\varphi)$. In particular E/E' is isomorphic to a subgroup of \mathbb{G}_a , namely $\tilde{E} := \varphi(E)$. Therefore, by the second part, it is enough to prove that $\text{Hom}(\tilde{E}, \mathbb{G}_a)$ is a free $\mathcal{U}[\delta_2]$ module of rank 1. But the map from $\text{Hom}(\tilde{E}, \mathbb{G}_a)$ to $\text{Hom}(E, \mathbb{G}_a)$ given by $\psi \mapsto \psi \circ \varphi$ is a module embedding. Thus $\text{Hom}(\tilde{E}, \mathbb{G}_a)$ is isomorphic to a submodule of $\mathcal{U}[\delta_2]$. This finishes the proof. \dashv

The following proposition gives a necessary condition for nonorthogonality between groups of the form E_f .

PROPOSITION 6.5. *Let f, g be in $\mathcal{U}[\delta_2]$. Suppose that E_f and E_g have nonorthogonal generics. Then f and g have the same order, say s . Moreover, if $s > 0$ then f and g have the same leading coefficient.*

PROOF. By 4.5 and 2.9 we know that f and g have the same order which we will denote by s . Suppose that $s > 0$. First we will look at a simple case in which nonorthogonality is witnessed by a homomorphism φ from E_f onto E_g . Since any homomorphism from E_f to \mathbb{G}_a can be expressed uniquely by an element in $\mathcal{U}[\delta_2]$, there are a natural number d and a_0, \dots, a_d in \mathcal{U} with $a_d \neq 0$ such that $\varphi(y)$ is given by $\sum_{i=0}^d a_i \delta_2^i y$. Let η be a generic element of E_f over the coefficients of φ, f and g . Since $\varphi(\eta)$ is in E_g we have

$$g(\varphi(\eta)) = \delta_1(\varphi(\eta)) = \varphi^{\delta_1}(\eta) + \varphi(\delta_1(\eta)) = \varphi^{\delta_1}(\eta) + \varphi(f(\eta)).$$

No nonzero element of $\mathcal{U}[\delta_2]$ contains E_f in its kernel, therefore this equality has to be a polynomial identity. So we can compare the coefficients of $\delta_2^{s+d}(\eta)$ on both sides. Let b_1 and b_2 be the leading coefficients of f and g , respectively. Since $s > 0$, $\varphi^{\delta_1}(\eta)$ does not have a term with $\delta_2^{s+d}(\eta)$. So on the left hand side we have $b_2 a_d$ and on the right hand side we have $b_1 a_d$ in front of $\delta_2^{s+d}(\eta)$. Therefore $b_1 = b_2$.

Now we can deal with the general case. As E_f and E_g have nonorthogonal generics, by 2.9, there are proper subgroups E'_f and E'_g of E_f and E_g , respectively, such that E_f/E'_f is isomorphic to E_g/E'_g . By 6.4, there is an f' in $\mathcal{U}[\delta_2]$ such that $E_{f'}$ is isomorphic to E_f/E'_f . Note that $E_{f'}$ is a homomorphic image of both E_f and E_g . Thus, by the previous case, f, g and f' all have same leading coefficient. \dashv

§7. Geometry of the heat variety. In this section we will work entirely in DCF_{02} . Let ζ be a fixed element of \mathcal{U} . Throughout this section, G will denote the group defined by $\delta_1 y = \delta_2^2 y + \zeta y$. So, if ζ is zero then G is the heat variety. We fix a small submodel K of \mathcal{U} that contains ζ and name all its elements. Let p be the generic type of G over K . The main goal of this section is to prove that p is locally modular. We define h to be the operator given by $\delta_1 - \delta_2^2$. Clearly, for $a, b \in \mathcal{U}$ we have

$$h(ab) = ah(b) + h(a)b - 2\delta_2(a)\delta_2(b).$$

If δ is a derivation, $\ell\delta$ is the logarithmic derivative of δ , i.e. $\ell\delta(y) = \delta(y)/y$.

From now on we will mainly deal with groups defined by equations of the form $h(y) = a\delta_2(y) + by$ for $a, b \in \mathcal{U}$. Note that such a group can also be described as E_f where $f = \delta_2^2 + a\delta_2 + b$. First we will prove a lemma explaining the behavior of such groups under certain maps.

LEMMA 7.1. *Let H be the group defined by $h(y) = a\delta_2(y) + by$ for some a, b in \mathcal{U} .*

1. *For $c \in \mathcal{U}^\times$, $c^{-1}H$ is defined by*

$$h(y) = (a + 2\ell\delta_2(c))\delta_2(y) + (b + a\ell\delta_2(c) - h(c)/c)y.$$

2. If $b = 0$ then $\delta_2(H)$ is defined by

$$\mathfrak{h}(y) = a\delta_2(y) + \delta_2(a)y.$$

PROOF. For the first part, fix a generic element η of $c^{-1}H$. Then $c\eta$ is a generic element of H and hence we have $\mathfrak{h}(c\eta) = a\delta_2(c\eta) + bc\eta$. This gives us

$$\mathfrak{h}(c\eta) + c\mathfrak{h}(\eta) - 2\delta_2(c)\delta_2(\eta) = a\delta_2(c)\eta + ac\delta_2(\eta) + bc\eta.$$

If we rearrange the terms, we see that $c^{-1}H$ is contained in the group defined by

$$\mathfrak{h}(y) = (a + 2\delta_2(c))\delta_2(y) + (b + a\delta_2(c) - \mathfrak{h}(c)/c)y$$

Since this equation defines a Δ -type minimal group, it defines $c^{-1}H$. For the second part, suppose η is a generic element of H . Then we have $\mathfrak{h}(\eta) = a\delta_2(\eta)$. By applying δ_2 to both sides we obtain

$$\mathfrak{h}(\delta_2(\eta)) = \delta_2(\eta)\delta_2(a) + a\delta_2^2(\eta).$$

The rest follows as in the first part. ◻

The next proposition is a useful finiteness result. Let H be the group defined by $\mathfrak{h}(y) = by$. For s a natural number, define

$$\mathrm{Hom}(H, G)_s = \{(a_0, \dots, a_s) \in \mathbb{G}_a^{s+1} : \sum_{i=0}^s a_i \delta_2^i(\eta) \in G \text{ for all } \eta \in H\}.$$

By convention $\mathrm{Hom}(H, G)_{-1}$ is 0. Clearly $\mathrm{Hom}(H, G)_s$ is a subgroup of \mathbb{G}_a^{s+1} defined over b . We will identify the tuples in $\mathrm{Hom}(H, G)_s$ and the morphisms they define.

PROPOSITION 7.2. *For all s , $\mathrm{Hom}(H, G)_s$ is a finite dimensional vector space over the constants.*

PROOF. First note that for $s \geq 0$ there is a homomorphism λ_s from $\mathrm{Hom}(H, G)_s$ to \mathbb{G}_a which maps (a_0, \dots, a_s) to a_s . Clearly $\ker(\lambda_s) = \mathrm{Hom}(H, G)_{s-1}$. The key observation here is the following: the image of λ_s is contained in $\ker(\delta_2)$. Indeed, if $\varphi(y) = \sum_{i=0}^s a_i \delta_2^i(y)$ and η is an element of H , generic over b, a_0, \dots, a_s , we have

$$\begin{aligned} \zeta\eta &= \mathfrak{h}(\varphi(\eta)) \\ &= \varphi^{\mathfrak{h}}(\eta) + \varphi(\mathfrak{h}(\eta)) - 2\varphi^{\delta_2}(\delta_2(\eta)) \\ &= \varphi^{\mathfrak{h}}(\eta) + \varphi(b\eta) - 2\varphi^{\delta_2}(\delta_2(\eta)). \end{aligned}$$

As in the proof of 6.5, this has to be a polynomial identity. Hence the coefficient of $\delta_2^{s+1}(\eta)$, which is $-2\delta_2(a_s)$, should be zero.

Suppose that our claim is false. Then there is a minimum $s \neq -1$ such that $\mathrm{Hom}(H, G)_s$ is infinite dimensional. Since s is minimal, $\mathrm{Hom}(H, G)_{s-1}$ is finite dimensional. Therefore the image of λ_s is infinite dimensional. This means that the image of λ_s is the field of δ_2 -constants. Now we can

use proposition 6.1 to find a rank ω subgroup \mathcal{H} of $\text{Hom}(H, G)_s$. Note that the typical Δ -dimension of \mathcal{H} has to be 1. Fix an element η in H such that $\varphi_0(\eta) \neq 0$ for some $\varphi_0 \in \mathcal{H}$. The map from \mathcal{H} to G defined by $\varphi \mapsto \varphi(\eta)$ is surjective since its image is nonzero and both \mathcal{H} and G are Δ -type minimal. But this contradicts 2.9 since \mathcal{H} and G have different typical Δ -dimensions. \dashv

Now we can prove our main theorem.

THEOREM 7.3. *Let H be the group defined by $\mathfrak{h}(y) = by$. Suppose that*

$$\varphi(y) = \sum_{i=0}^s a_i \delta_2^i(y) \quad , \quad a_s \neq 0$$

defines a homomorphism from H onto G . Then $\text{stp}(a_0, \dots, a_s, b)$ is p -simple of p -weight zero.

Recall that we identify the tuple of coefficients of a linear δ_2 -polynomial with the morphism it defines. So $\text{stp}(\varphi, b)$ means $\text{stp}(a_0, \dots, a_s, b)$.

PROOF. First we will show that it is enough to prove the theorem in the case of $a_s = 1$. From the proof of proposition 7.2 we know that $\delta_2(a_s) = 0$. So, by looking at the first part of 7.1, we see that $a_s H$ is defined by $\mathfrak{h}(y) = (b - \mathfrak{h}(a_s^{-1})a_s)y$. Moreover, the map from $a_s H$ to G defined by $\varphi'(y) := \varphi(a_s^{-1}y)$ has leading coefficient 1. Suppose that $\text{stp}(\varphi', b - \mathfrak{h}(a_s^{-1})a_s)$ is p -simple of p -weight zero. Note that (φ, b) is in the definable closure of $(\varphi', b - \mathfrak{h}(a_s^{-1})a_s, a_s)$. As a_s is a δ_2 -constant, it is p -simple of p -weight 0. Therefore $\text{stp}(\varphi, b)$ is p -simple and we have

$$\begin{aligned} w_p(\varphi, b) &\leq w_p(\varphi', b - \mathfrak{h}(a_s^{-1})a_s, a_s) \\ &= w_p(\varphi', b - \mathfrak{h}(a_s^{-1})a_s/a_s) + w_p(a_s) = 0. \end{aligned}$$

So we can assume that $a_s = 1$. We will prove the statement by induction on s . If $s = 0$ then φ is identity since its leading coefficient is 1. So $b = 0$ and we are done. Now suppose $s > 0$. We will factor φ .

First thing we need to show is that φ has a nontrivial kernel. Suppose not. Then φ is an isomorphism from H onto G . Let φ^{-1} be its inverse. Let us express φ^{-1} by an element of $\mathcal{U}[\delta_2]$. Now $\varphi \circ \varphi^{-1}(y) - y$ is an element of $\mathcal{U}[\delta_2]$ that vanishes on G . So it must be the zero map. Hence $\varphi(y)$ is of order zero. Contradiction.

Let α be a nonzero element in the kernel of φ . Then, by 7.1, $\alpha^{-1}H$ is defined by the equation

$$\mathfrak{h}(y) = 2\ell\delta_2(\alpha)\delta_2(y)$$

since $\mathfrak{h}(\alpha)/\alpha = b$. Let $\psi_0(y) = \varphi(\alpha y)$. Clearly ψ_0 defines a homomorphism from $\alpha^{-1}H$ to G . Moreover $\psi_0(1) = 0$, which implies that there is an element in $\mathcal{U}[\delta_2]$ such that $\psi_0(y) = \psi_1(\delta_2(y))$. Similarly ψ_1 defines

a map from $\delta_2(\alpha^{-1}H)$ to G . By the second part of 7.1, we know that $\delta_2(\alpha^{-1}H)$ is defined by

$$\mathfrak{h}(y) = 2\ell\delta_2(\alpha)\delta_2(y) + 2\delta_2(\ell\delta_2(\alpha))y.$$

Hence $H' := \alpha\delta_2(\alpha^{-1}H)$ is defined by $\mathfrak{h}(y) = b'y$ where

$$b' = 2\delta_2(\ell\delta_2(\alpha)) - 2\ell\delta_2(\alpha)^2 - \mathfrak{h}(\alpha^{-1})\alpha.$$

Let $\psi_2(y)$ be the map from H' to G given by $\psi_1(\alpha^{-1}y)$. Now we will factor φ and apply induction. We have

$$\begin{aligned} \varphi(y) &= \psi_0(\alpha^{-1}y) \\ &= \psi_1(\delta_2(\alpha^{-1}y)) \\ &= \psi_2(\alpha\delta_2(\alpha^{-1}y)) \\ &= \psi_2(\delta_2(y) - \ell\delta_2(\alpha)y). \end{aligned}$$

We will change the notation here and denote ψ_2 by φ' . Since both φ and $\delta_2(y) - \ell\delta_2(\alpha)y$ have leading coefficient 1, so does φ' . Moreover φ' is a homomorphism from H' to G and the order of φ' is less than the order of φ . So, by induction, $\text{stp}(\varphi', b')$ is p -simple of p -weight 0.

In order to finish the proof, we need to know the relation between (φ, b) and (φ', b') . Let $\beta = -\ell\delta_2(\alpha)$. We know that $\delta_2(y) + \beta y$ defines a homomorphism from H to H' . Let $\eta \in H$ be generic over b, b' and β . Then we have $\mathfrak{h}(\delta_2(\eta) + \beta\eta) = b'(\delta_2(\eta) + \beta\eta)$. By using the fact that $\mathfrak{h}(\eta) = b\eta$ we obtain

$$\delta_2(b)\eta + b\delta_2(\eta) + \mathfrak{h}(\beta)\eta + \beta b\eta - 2\delta(\beta)\delta_2(\eta) = b'\delta_2(\eta) + b'\beta\eta.$$

By looking at the coefficient of η we get

$$\delta_2(b) + \mathfrak{h}(\beta) + \beta b = \beta b',$$

and looking at the coefficient of $\delta_2(\eta)$ we get

$$b = b' + 2\delta_2(\beta).$$

We will show that $\text{stp}(\beta)$ is p -simple of p -weight 0. By eliminating b from the two equations above, we get

$$\delta_2(b') + 2\delta_2^2(\beta) + \mathfrak{h}(\beta) + 2\beta\delta_2(\beta) = 0.$$

Since this equation is of the form $\delta_1(y) = f(y)$ for some (not linear) δ_2 -polynomial f with coefficients in $\text{dcl}(b')$, any type containing it is necessarily of Morley rank at most ω . Thus, $q := \text{stp}(\beta/b')$ is either of finite rank or it is of rank ω . Hence, in any case q is p -simple. If $\text{RU}(q) < \omega$ then clearly q is of p -weight 0. So suppose $\text{RU}(q) = \omega$. Note that it is enough to show that p and $\text{stp}(q)$ are orthogonal. But if these types are not orthogonal, then p and the generic type of the tangent space of $\text{loc}(\beta/b')$ at a suitable point are nonorthogonal. Clearly, the generic type of the tangent space of $\text{loc}(q)$ at any point contains a formula of the form

$\delta_1(y) = -\delta_2^2 y + \nu_1 \delta_2 y + \nu_0 y$ for some ν_1, ν_0 in \mathcal{U} . Then the tangent space is defined by $\delta_1(y) = -\delta_2^2 y + \nu_1 \delta_2 y + \nu_0 y$ since it is of infinite rank. By proposition 6.5, this is impossible. Therefore we proved that q is p -simple of p -weight 0. As $\text{stp}(b')$ is p -simple of p -weight 0 by induction, $\text{stp}(\beta)$ is p -simple of p -weight zero, too.

The strong type of b over the empty set is p -simple since $b = b' + 2\delta_2(\beta)$ and the pair (β, b') is p -simple of p -weight zero. Since $\varphi(y) = \varphi'(\delta_2(y) - \beta y)$, φ (viewed as a tuple) is in the definable closure of (β, φ') . So the tuple (φ, b) is p -simple and we can talk about the p -weight of it. By proposition 7.2, $\text{RU}(\varphi/b) < \omega$. Hence $w_p(\varphi, b) = w_p(\varphi/b) + w_p(b) = w_p(b)$. So we are done as b is in the definable closure of (β, b') and $w_p(\beta, b') = w_p(\beta/b') + w_p(b') = 0$.

⊖

This theorem has the following corollary.

THEOREM 7.4. *Fix an element ζ of \mathcal{U} and a small submodel K of \mathcal{U} that contains ζ . Let G be the subgroup of \mathbb{G}_a defined by $\mathfrak{h}(y) = \zeta y$ and let p be its generic type over K .*

1. *If H is a p -semiregular subgroup of $G \times G$ of p -weight 1 then its canonical parameter has p -weight 0.*
2. *p is locally modular.*

PROOF. First note that H is Δ -type minimal by proposition 6.1. We will show that $\text{Hom}(H, \mathbb{G}_a)$ is a free $\mathcal{U}[\delta_2]$ -module of rank 1. Clearly the restriction map from $\text{Hom}(G \times G, \mathbb{G}_a) = \mathcal{U}[\delta_2]y_1 \oplus \mathcal{U}[\delta_2]y_2$ to $\text{Hom}(H, \mathbb{G}_a)$ is surjective and has a nontrivial kernel. So the $\mathcal{U}[\delta_2]$ -rank of the free part of $\text{Hom}(H, \mathbb{G}_a)$ is 1. If $\varphi \in \text{Hom}(H, \mathbb{G}_a)$ is torsion, then $\varphi(H)$ is contained in a finite dimensional vector space over the δ_2 -constants. But H is orthogonal to field types by 3.3 and 2.9. Therefore $\text{Hom}(H, \mathbb{G}_a)$ is torsion-free, which means that there is a $\psi \in \text{Hom}(H, \mathbb{G}_a)$ such that $\text{Hom}(H, \mathbb{G}_a) = \mathcal{U}[\delta_2]\psi$.

Choose $f_1, f_2 \in \mathcal{U}[\delta_2]$ such that $f_i \psi$ is the i -th projection from H to G for $i = 1, 2$. If (η_1, η_2) is in the kernel of ψ , then

$$\eta_i = f_i(\psi(\eta_1, \eta_2)) = 0$$

for $i = 1, 2$. Hence ψ is injective. Let $H' = \psi(H)$ and let $\varphi_1, \varphi_2 \in \text{Hom}(H', G)$ be such that $\psi^{-1}(y) = (\varphi_1(y), \varphi_2(y))$. Since $\text{Hom}(H', \mathbb{G}_a)$ is $\mathcal{U}[\delta_2]$, H' is defined by an equation of the form $\delta_1(y) = f(y)$ where $f \in \mathcal{U}[\delta_2]$. Since the generic type of H' is nonorthogonal to the generic type of G , by proposition 6.5, $f(y) = \delta_2^2(y) + a\delta_2(y) + by$ for some $a, b \in \mathcal{U}$. Replacing H' by $c^{-1}H'$ for some c with $\ell\delta_2(c) = -a/2$, we can assume that $a = 0$ by (*). So, H' is defined by $\mathfrak{h}(y) = by$. But this finishes the proof of the first part because H is defined over b, φ_1, φ_2 and these have p -weight 0 by theorem 7.3. The second part follows from 6.3. ⊖

§8. An application to algebra. The definition of a Jordan-Hölder series for abelian differential algebraic groups is a problematic issue since the notion of simplicity is problematic. The following definition, which was formulated in the context of pseudogroups in [1], is due to Elie Cartan.

DEFINITION 8.1. A Δ -subgroup G of \mathbb{G}_a is simple improperly speaking if every non-trivial homomorphic image of G is isomorphic to G .

Inspired by the work of Cartan, Cassidy, in her talk in *Leeds Differential Fields Meeting* in 2007, defined a Jordan-Hölder series to be a normal series whose quotients are simple improperly speaking, and asked if the heat variety admits such a series. Here we show that the answer is negative.

THEOREM 8.2. *Let G be a nontrivial quotient of the heat variety. Then there is a homomorphism φ from G to \mathbb{G}_a such that there is no homomorphism from $\varphi(G)$ onto G . In particular, the heat variety does not admit a Jordan-Hölder series whose quotients are simple improperly speaking.*

PROOF. First of all, note that any nontrivial quotient of the heat variety is isomorphic to a group of the form E_f for some f in $\mathcal{U}[\delta_2]$ by the third part of 6.4. Moreover, since G and E_f are nonorthogonal, by 6.5, f should be of the form $\delta_2^2 + a\delta_2 + b$ for some a, b in \mathcal{U} . So E_f is defined by the equation $h(y) = a\delta_2(y) + by$. By replacing E_f by $c^{-1}E_f$ for some c in \mathcal{U} with $\ell\delta(c) = -a/2$, by 7.1, we can assume that $a = 0$. We have just proved that any nontrivial quotient of the heat variety is isomorphic to a group defined by $h(y) = by$ for some b in \mathcal{U} . Therefore, it suffices to show that no such group is simple improperly speaking.

Suppose that G is a group defined by the equation $h(y) = by$. Fix a generic element η of G over b . Now consider the differential operator

$$\varphi(y) := \eta\delta_2(\eta^{-1}y) = \delta_2(y) - \ell\delta_2(\eta)y.$$

By using 7.1 as in the proof of 7.3, we see that $\varphi(G)$ is defined by the equation $h(y) = b'y$ where

$$b' = \delta_2(2\ell\delta_2(\eta)) - 2\ell\delta_2(\eta)^2 - \eta h(\eta^{-1}).$$

By 7.3, if there is a homomorphism from $\varphi(G)$ to G , then $w_p(b') = 0$ where p is the generic type of G over b' . However, b' is of the form $f(\eta)$ for some nontrivial δ_2 -rational function f , namely

$$f(y) = \frac{2\delta_2^2(y)y - 3\delta_2(y)^2 + by^2}{y^2}.$$

This shows that $w_p(b') = 1$. Contradiction. -1

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